1 Complementary Slackness

Given a Primal system, we will also look at its Dual system:

<table>
<thead>
<tr>
<th>Primal $P$</th>
<th>Dual $D$</th>
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| $\min \sum_{j=1}^{n} c_j x_j$, so that: \begin{align*} 
\forall i & \ (1 \leq i \leq m) \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i \\
\forall j & \ (1 \leq j \leq n) \quad x_j \geq 0 \end{align*} | $\max \sum_{i=1}^{m} b_i y_i$, so that: \begin{align*} 
\forall j & \ (1 \leq j \leq n) \quad \sum_{i=1}^{m} a_{ij} y_i \leq c_j \\
\forall i & \ (1 \leq i \leq m) \quad y_i \geq 0 \end{align*} |

**Theorem 1 (Complementary Slackness Principal)**

If $x, y$ are solutions to $P, D$, respectively, and $\alpha, \beta \geq 1$ fulfill the following conditions:

1. $\forall j : x_j = 0$ or $\frac{c_j}{\alpha} \leq \sum_i a_{ji} y_i \leq c_j$
2. $\forall i : y_i = 0$ or $b_i \leq \sum_j a_{ji} x_j \leq \beta b_i$

Then: $\sum_{j=1}^{n} c_j x_j \leq \alpha \beta \sum_{i=1}^{m} b_i y_i$.

Let us look at the LP system for the SET-COVER problem, where $U$ is the set of elements, $S \subseteq P(U)$, and for each set $s \in S$ we mark $c(s)$ to be the cost of set $s$.

- $\min \sum_{s \in S} c(s) x_s$ so that:
  - $\forall e \in U : \sum_{s|e \in s} x_s \geq 1$
  - $\forall s \in S : x_s \geq 0$

Let us look at its dual system:

- $\max \sum_{e \in U} y_e$ so that:
  - $\forall s \in S : \sum_{e|e \in s} y_e \leq c(s)$
  - $\forall e \in U : x_e \geq 0$

We shall see an $f$-approximation algorithm for SET-COVER.
2 SET-COVER Algorithm

*Primal condition:* $\alpha = 1$.
Hence $\forall s \in S, x_s \neq 0 \Rightarrow \sum_{e \in s} y_e = c(s)$

*Dual condition:* $\beta = f$.
Hence $\forall e \in U, y_e \neq 0 \Rightarrow \sum_{s | e \in s} x_s \leq f$

As long as there is an element $e$ which is not yet covered:

- Increase $y_e$ until the set $s$ is tight.
- Add $s$ to the cover ALG.
- Remove from $U$ all the elements covered by $s$.

**Claim 2** *Primal solution ALG is a legal solution.*

*Proof:* Because for each uncovered $e \in U$, the algorithm adds a set $s$ which covers it to the cover.

**Claim 3** *The Dual solution is also legal.*

*Proof:* Because we always increase a $y_e$ only until the first set $s \in S$ is tight, never beyond $c(s)$, and so the condition $\sum_{e | e \in s} y_e \leq c(s)$ is preserved.

**Claim 4** *The Primal condition is met.*

*Proof:* Because for any $s$ in the cover (i.e. $x_s \neq 0$) we always increased $y_e$ until the inequality was tight, hence turning it into an equation as necessary.

**Claim 5** *The Dual condition is also met.*

*Proof:* Obviously the number of sets any element $e$ is in must be smaller or equal to the frequency $f$, by the definition of frequency.

**Conclusion:** From the Complementary Slackness Theorem, we have proved $f$-approximation.
3 MIN-MULTICUT

We are given a graph $G=(V,E)$, and a group $K = \{(s_1,t_1),..., (s_k,t_k)\}$, where $s_i,t_i \in V$. We define $k = |K|$.
For every $e \in E$, we define a weight $c_e$.

**Goal:** To find a cut of minimal weight so that each pair of vertices $(s_i,t_i) \in K$ is separated by the cut.
This is a very difficult problem, even under the assumption (which we shall make) that $G$ is a tree.

We will define variables. For each $e \in E$, $0 \leq d_e \leq 1$ represents whether or not $e$ is in the cut.
For each $i$, we will mark as $P_i$ the path between $s_i$ and $t_i$. (Note: $P_i$ is uniquely defined, as $G$ is a tree.)
We will reach primal and dual systems as follows:

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<td>$\min \sum_{e \in E} c_e d_e$, so that: $\forall i : (1 \leq i \leq k)$ $\sum_{e \in P_i} d_e \geq 1$ $\forall j$ $d_e \geq 0$</td>
<td>$\max \sum_{i=1}^k f_i$, so that: $\forall e$ $\sum_{i</td>
</tr>
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We note that the dual system is precisely the LP program for finding maximal flow in a tree graph.

4 MIN-MULTICUT Algorithm

**Primal condition:** $\alpha = 1$.
Hence $\forall e \in E$, $d_e \neq 0 \Rightarrow \sum_{i|e \in P_i} f_i = c_e$
In other words, each edge in the multicut is at maximal flow.

**Dual condition:** $\beta = 2$.
Hence $\forall i$, $f_i \neq 0 \Rightarrow \sum_{e \in P_i} d_e \leq 2$
In other words, in a path with nonzero flow, at most 2 edges may be added to the cut.

**Initialization:** We begin with a group $D = \emptyset$, and we assume $\forall i : f_i = 0$.
We choose a vertex $v$, and mark it as our root. We define the depth of the root to be 0.
For any vertex $u$, we define $u$’s depth to be $u$’s distance from the root $v$.
We will define the $lca$ (least common ancestor) of two vertices $s_i, t_i$ to be the vertex with the smallest depth value on the path $P_i$, and we will mark this vertex $lca(s_i,t_i)$.
Flow: For every vertex v in the graph, in order of nonascending depth (i.e. deepest vertices first):
If there exist an i so that $v = lca(s_i, t_i)$, then we greedily increase the flow from $s_i$ to $t_i$
(i.e., raise the flow as much as the path’s capacity will allow).
We now add to D all edges which have reached their maximal flow during this iteration.
We will mark the final result as $D = \{e_1, ... e_l\}$.

Backwards Deletion: For every $1 \leq j \leq l$:
If $D \setminus \{e_j\}$ is a multicut, remove $e_j$ from D.

**Claim 6** The primal solution is a legal one.

**Proof:** On every path $P_i$, there is a sated edge which is now in the cut, else we’d have kept increasing the flow on that path.

**Claim 7** The dual solution is a legal one.

**Proof:** We never exceeded the capacity of any edges.

**Claim 8** The primal condition is met.

**Proof:** We chose only the sated edges (i.e. the ones meeting this condition) for the cut.

**Claim 9** The dual condition is met.

**Proof:** next tirgul.

**Conclusion:** By the Complementary Slackness Theorem, we have proved 2-approximation.