THE INFLUENCE OF VARIABLES IN PRODUCT SPACES

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ABSTRACT

Let $X$ be a probability space and let $f: X^n \to \{0, 1\}$ be a measurable map. Define the influence of the $k$-th variable on $f$, denoted by $I_f(k)$, as follows: For $u = (u_1, u_2, \ldots, u_n) \in X^{n-1}$ consider the set $I_k(u) = \{(u_1, u_2, \ldots, u_{k-1}, t, u_k, \ldots, u_n) : t \in X\}$.

$$I_f(k) = Pr(u \in X^{n-1} : f \text{ is not constant on } I_k(u)).$$

More generally, for $S$ a subset of $[n] = \{1, \ldots, n\}$ let the influence of $S$ on $f$, denoted by $I_f(S)$, be the probability that assigning values to the variables not in $S$ at random, the value of $f$ is undetermined.

THEOREM 1: There is an absolute constant $c_1$ so that for every function $f: X^n \to \{0, 1\}$, with $Pr(f^{-1}(1)) = p \leq \frac{1}{2}$, there is a variable $k$ so that

$$I_f(k) \geq c_1 p \log \frac{n}{p}$$

THEOREM 2: For every $f: X^n \to \{0, 1\}$, with $Pr(f = 1) = \frac{1}{2}$, and every $\epsilon > 0$, there is $S \subseteq [n], |S| = c_2(\epsilon)n/\log n$ so that $I_f(S) \geq 1 - \epsilon$.

These extend previous results by Kahn, Kalai and Linial for Boolean functions, i.e., the case $X = \{0, 1\}$.

1. Introduction

Let $X$ be a probability space and let $f: X^n \to \{0, 1\}$ be a measurable map. Define the influence of the $k$-th variable on $f$, denoted by $I_f(k)$, as follows: For $u = (u_1, u_2, \ldots, u_{n-1}) \in X^{n-1}$ consider the set

$$I_k(u) = \{(u_1, u_2, \ldots, u_{k-1}, t, u_k, \ldots, u_n) : t \in X\}.$$

(1) $$I_f(k) = Pr(u \in X^{n-1} : f \text{ is not constant on } I_k(u)).$$

More generally, for $S$ a subset of $[n] = \{1, \ldots, n\}$ let the influence of $S$ on $f$, denoted by $I_f(S)$, be the probability that assigning values to the variables not in $S$ at random, the value of $f$ is undetermined. (Note that $I_f(\{k\}) = I_f(k)$.)

The purpose of this note is to supplement the papers by Kahn, Kalai and Linial [KKN, KKL], which study the influence of variables on Boolean functions, i.e., the case $X = \{0, 1\}$. The reader is referred to [EL, KKL, KKL'] for background on this problem and its relevance to extremal combinatorics and theoretical computer science.

Given $X$ and $f$ as above we can replace $X$ by the unit interval $[0, 1]$, and $f$ by an appropriate function $g$ so that the influences of $f$ and $g$ will be the same. Therefore, there will be no loss of generality in assuming that $X = [0, 1]$.

An easy consequence of Loomis and Whitney's inequality [LW] is:

THEOREM 0: Every function $f: X^n \to \{0, 1\}$ with $Pr(f = 1) = p \leq \frac{1}{2}$ satisfies

$$\sum_{k=1}^{n} I_f(k) \geq p \log \frac{1}{p}.$$

The following examples show that for $p > \left(\frac{1}{2}\right)^n$ this inequality is sharp (up to a constant factor): If $(\frac{1}{2})^{k-1} \geq p > (\frac{1}{2})^k$ let $f = 1$ if $p^{1/k} \geq x_i$, for $i = 1, \ldots, k$.

Theorem 0 implies that for some variable $k$,

$$I_f(k) \geq p \log \frac{1}{p} \frac{1}{n}.$$

Here we improve this estimate to

THEOREM 1: There is an absolute constant $c_1$ so that for every function

$$f: X^n \to \{0, 1\},$$

with $Pr(f = 1) = p \leq \frac{1}{2}$, there is a variable $k$ so that

$$I_f(k) \geq c_1 \frac{\log n}{n}.$$

Repeated applications of Theorem 1 yields:

THEOREM 2: For every $f: X^n \to \{0, 1\}$, with $Pr(f = 1) = \frac{1}{2}$, and every $\epsilon > 0$, there is $S \subseteq [n], |S| = c_2(\epsilon)n/\log n$ so that $I_f(S) \geq 1 - \epsilon$.

The assertions of Theorems 1 and 2 for Boolean functions (i.e., for the special case $X = \{0, 1\}$) are proved in [KKN, KKL], in response to a conjecture by Ben-Or and Linial [BL], that Theorems 1 and 2 are asymptotically optimal for $p = \frac{1}{2}$ and $X = \{0, 1\}$ is shown by the "tribes" function $f$ from [BL]. Here, and throughout the paper, we identify elements of $\{0, 1\}^n$ with subsets $S$ of $[n]$ in the usual way. Partition $[n]$ into subsets $S_1, \ldots, S_k$ of size $\log n - \log \log n + c$
(c is an appropriate constant) and define \( f(T) = 1 \) iff \( T \) contains \( S_j \) for some \( j \). Obviously, a similar function can also be realized for \( X = [0, 1] \).

An example which exists only in the latter case but not for \( X = \{0, 1\} \) is the function \( f \) which equals 1 iff \( x_i \leq p^{1/n} \) for every \( i \), \( 1 \leq i \leq n \). It shows the Loomis–Whitney inequality to be tight for any \( p > 0 \) and also shows why the proof in [KKL, KKL'] needs to be modified to handle general probability spaces \( X \).

2. Proofs

The proof of [KKL] relies on Beckner’s hypercontractive estimate. In order to extend it to our more general case we need some additional considerations. We also sketch a variant of the proof based on another hypercontractive estimate. For simplicity we prove Theorem 1 for \( p = \frac{1}{2} \), leaving the minor adjustment needed for general \( p \) to the reader.

**Lemma 1:** Given a function \( g : [0, 1]^n \to \{0, 1\} \), there is a monotone function \( f : [0, 1]^n \to \{0, 1\} \) such that \( I_f(k) \geq I_g(k) \) for every \( k \).

**Proof:** Consider the restriction of \( f \) to the unit segment \( l_k(u) \). Define \( T_k(f) \) as the function which is monotone on \( l_k(u) \) and satisfies \( \Pr(f(T_k(f)) = f(f^{-1}(0) \cap l_k(u)) = \Pr(f^{-1}(0) \cap l_k(u)) \) for every \( u \in X^{n-1} \). Note that \( I_f(k) = I_{T_k(f)}(k) \) and \( I_f(j) \geq I_{T_k(f)}(j) \) for \( j \neq k \). Repeated applications of these operations yields in a limit a function which is fixed under all \( T_k \), hence monotone.

**Remark 1:** The proof of Lemma 1 is a standard combinatorial shifting argument, (see [A, Bo, F, BL]) and is also similar to the well-known Steiner symmetrization.

**Remark 2:** The same argument implies that \( I_f(S) \geq I_g(S) \) for every \( S \).

At this point we replace \( X = [0, 1] \) by the interval of integers

\[
Y = \{0, 1, \ldots, 2^m - 1\}
\]

(with uniform probability distribution). It suffices to prove Theorem 1 with \( Y \) instead of \( X \) as long as our constants do not depend on \( m \). It will be useful to identify \( Y \) with the discrete \( m \)-dimensional cube \( [0, 1]^m \) by the binary expansion. This allows one to express functions \( f : Y \to \mathbb{R} \) in their Walsh–Fourier expansion

\[
f = \sum \{ \hat{f}(S) u_S : S \subseteq [m] \},
\]

where \( u_S \) is the function defined by \( u_S(T) = (-1)^{|S \cap T|} \).

For a function \( f : Y^n \to \mathbb{R} \), we write the Walsh–Fourier expansion of \( f \) in the following form:

\[
f = \sum \{ \hat{f}(S_1, \ldots, S_n) u_{S_1} \cdots u_{S_n} : |S_1| \leq m, \ldots, |S_n| \leq m \}.
\]

Here \( u_{S_1} \cdots u_{S_n} (T_1, \ldots, T_n) = \prod u_{S_i}(T_i) \).

We always view \( Y \) as a probability space, and so given a function \( f : Y \to \mathbb{R} \), its \( p \)-th norm is defined as

\[
\|f\|_p = \left( \frac{1}{|Y|} \sum_{S \subseteq [m]} |\hat{f}(S)|^p \right)^{1/p}.
\]

Parseval’s identity asserts that \( \|f\|_2^2 = \sum_{S \subseteq [m]} \hat{f}(S)^2 |S| \).

Clearly, \( w(f) \geq 0 \) for every function \( f \) and \( w(f) = 0 \) if and only if \( f \) is a constant function.

**Lemma 2:** ([KKL, CG]) For \( f : [0, 1]^m \to \{0, 1\} \)

\[
w(f) = \sum_{k=1}^{m} I_f(k).
\]

A function \( f \) from \( Y \) to \( \{0, 1\} \) is monotone iff for some \( t, f(i) = 0 \) when \( 0 \leq i < t \) and \( f(i) = 1 \) when \( t < i \leq 2^m - 1 \). This has some implications on \( f \)'s Walsh transform.

**Lemma 3:** Let \( f : Y \to \{0, 1\} \) be a monotone function. Then \( w(f) \leq 2 \).

**Proof:** By definition \( I_f(k) = 2^{-m+1} \) times the number of pairs \( u, w \) with \( f(u) = 0, f(w) = 1 \) so that \( v \) is obtained from \( w \) by flipping the \( k \)-th coordinate. (Note: here, \( I_f(k) \) is the influence of a function from \( Y \) to \( \{0, 1\} \) regarded as a Boolean function of \( m \) variables.)

The monotonicity of \( f \) implies that

\[
I_f(1) \leq \frac{1}{2^{(m-1)}}, \quad I_f(2) \leq \frac{1}{2^{(m-2)}}, \ldots, I_f(m) \leq 1
\]

(in fact,

\[
I_f(k) = \frac{1}{2^{(m-k)}}
\]

unless \( t < 2^{k-1} \) or \( t > 2^m - 2^{k-1} \). Therefore \( \sum_{k=1}^{m} I_f(k) \leq 2 \), and by Lemma 2 this is what we need.
LEMMA 4: ([KKL]) For $f: \{0,1\}^r \to \{0,1\}$, define $T_k(f) = \sum \{ f(S)e^{|S|} u_S : S \subseteq \{r\} \}$. Then

$$\|T_k f\|_2 \leq \|f\|_1 + \epsilon.$$  

Proof: As shown in [KKL] this follows at once from Lemmas 1 and 2 in Beckner’s paper [Be]. (We will need the case $r = mn$.) 

Remark: For our purposes $1 + \epsilon^2$ can be replaced by any $2 - \delta(\epsilon)$, so Beckner’s Lemma 1 can be replaced here by an obvious estimate.

Here is a quick outline of the proof of Theorem 1. We assume that $f$ is monotone. Consider the restriction $g$ of $f$ to a function from $Y$ to $\{0,1\}$ obtained by assigning values to all variables except the $k$-th one. $I_f(k)$ is the probability (assignments being selected at random) that $g$ is not constant. The proof is based on two observations: First, that $w(g)$ is bounded between 0 and 2 with $w(g) = 0$ if $g$ is constant. The second observation is that if $r$ is obtained by subtracting from $g$ its average value, then $r$ is bounded, and we can give an absolute upper bound for the $(4/3)$-norm of $r$. These two observations combined with Lemma 4 have consequences on the Walsh–Fourier coefficients of $f$ which imply our theorem.

Proof of Theorem 1: Let $f: Y^n \to \{0,1\}$ be a function with $Pr(f = 1) = \frac{1}{2}$. We will show that for some $k$,

$$I_f(k) \geq c_1 \frac{\log n}{n}.$$  

By Lemma 1 we may assume that $f$ is monotone.

Let $T: Y \to R$ be given by $T(Z) = \sum g u_S(Z)|S|^{1/2}$, i.e. $\hat{T}(S) = |S|^{1/2}$ for all $S$. The convolution of $T$ with a function $g: Y \to R$ is denoted $T \ast g$, i.e., $\hat{T \ast g}(S) = \hat{g}(S)|S|^{1/2}$ and

$$\|T \ast g\|_2^2 = \sum_{S \subseteq \{m\}} \hat{g}^2(S)|S| = w(g).$$

Fix an index $n \geq k \geq 1$, and define a function

$$g = g[S_1, ..., S_{k-1}, S_{k+1}, ..., S_n]: Y \to \{0,1\}$$

by

$$g[S_1, ..., S_{k-1}, S_{k+1}, ..., S_n](S) = f(S_1, S_2, ..., S_{k-1}, S, S_{k+1}, ..., S_n).$$

(10) $v[S_1, S_2, ..., S_{k-1}, S_{k+1}, ..., S_n] = T \ast g[S_1, ..., S_{k-1}, S_{k+1}, ..., S_n]$. 

By equation (8), $\|v\|_2^2 = w(g)$, and by Lemma 3, $0 \leq \|v\|_2^2 \leq 2$. If $g$ is a constant function then $\|v\|_2^2 = 0$. Define now $W_k(S_1, S_2, ..., S_n) = v[S_1, ..., S_{k-1}, S_{k+1}, ..., S_n](S_k)$. $W_k$ is the convolution of $f$ with the real function $T_k$ on $Y^n$ given by $T_k(S_1, S_2, ..., S_n) = T(S_k)$ if $S_i = 0$ for every $i \neq k$ and $T_k(S_1, S_2, ..., S_n) = 0$ otherwise. Note that

$$\hat{T}_k(S_1, S_2, ..., S_n) = |S_k|^{1/2}$$

and

$$\|W_k\|_2^2 = \sum (\hat{W}_k(S_1, S_2, ..., S_n))^2 = \sum \hat{f}^2(S_1, S_2, ..., S_n)|S_k|.$$  

(11) $\hat{W}_k(S_1, S_2, ..., S_n) = \hat{f}(S_1, S_2, ..., S_n)|S_k|^{1/2},$

and

(12) $\|W_k\|_2^2 = \sum (\hat{W}_k(S_1, S_2, ..., S_n))^2 = \sum \hat{f}^2(S_1, S_2, ..., S_n)|S_k|.$

On the other hand,

$$\|W_k\|_2^2 = \|Y\|^{-n} \sum_{S_1 \subseteq \{m\}, ..., S_n \subseteq \{m\}} v[S_1, ..., S_{k-1}, S_{k+1}, ..., S_n]^2(S_k)$$  

(13) $\|W_k\|_2^2 = \|Y\|^{-n+1} \sum_{S_1, S_2, ..., S_{k-1}, S_{k+1}, ..., S_n} \|v[S_1, ..., S_{k-1}, S_{k+1}, ..., S_n]\|_2^2.$

But we saw that the value of $\|v[S_1, ..., S_{k-1}, S_{k+1}, ..., S_n]\|_2^2$ is non-negative, bounded by 2, and is equal to zero if $g[S_1, ..., S_{k-1}, S_{k+1}, ..., S_n]$ is the constant function.

Therefore we have

(14) $\|W_k\|_2^2 \leq 2I_f(k).$

Assume now that for every $k$,

$$I_f(k) \leq c_1 \frac{\log n}{n}.$$  

It follows that

$$\sum \|W_k\|_2^2 = \sum \|W_k\|_2^2$$  

(15) $\leq 2c_1 \log n.$
Thus, more than half of the weight of $||f||_2$ is concentrated where $|S_1| + |S_2| + \ldots + |S_n| < 5c_1 \log n$.

To reach a contradiction write $R_k = \sum_{S_k \neq \emptyset} \hat{f}(S_1 \ldots S_n) u(S_1 \ldots S_n)$. Note that

$$R_k(S_1, \ldots, S_{k-1}, S_k, S_{k+1}, \ldots, S_n) = f(S_1, \ldots, S_k, S_{k+1}, \ldots, S_n) - E_{S_k} f(S_1, \ldots, S_k, \ldots, S_n).$$

(16)

Here, $E_{S_k} f(S_1, \ldots, S_k, \ldots, S_n)$ is the average value of $f(S_1, \ldots, S_k, \ldots, S_n)$ over all values of $S_k$. Therefore $|R_k|$ is bounded (say by 2), and $R_k(S_1, \ldots, S_n) = 0$ if $g[S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_n]$ is a constant function.

It follows that

$$||R_k||_{4/3} \leq 3I_f(k).$$

(17)

I.e.,

$$||R_k||_{4/3} \leq (3I_f(k))^{3/2},$$

(18)

and by Lemma 4 for $\epsilon = \sqrt{3}/3$

$$\sum_{k=1}^{n} ||T \epsilon R_k||_{3/2} \leq \sum_{k=1}^{n} ||R_k||_{4/3} \leq c_2 (\log n)^{3/2} n^{-1/2}.$$  

(19)

Note that

$$T \epsilon R_k = \sum_{S_k \neq \emptyset} \hat{f}(S_1, \ldots, S_n) e^{i(S_1+\ldots+S_n)u(S_1 \ldots S_n)}$$

and that $\hat{f}(S_1, \ldots, S_n) = 0$ or $\hat{f}(S_1, \ldots, S_n)$, depending on $S_k$ being empty or not. Therefore,

$$\sum_{S_1 \subset \{1, \ldots, n\}} \sum_{S_2 \subset \{1, \ldots, n\}} \ldots \sum_{S_n \subset \{1, \ldots, n\}} \hat{f}(S_1 \ldots S_n) \mu(S_1 \ldots S_n) e^{i(S_1+\ldots+S_n)} \leq c_3 (\log n)^{3/2} n^{-1/2},$$

(20)

where $\mu(S_1 \ldots S_n) = |\{i : S_i \neq \emptyset\}|$.

The last relation implies that more than half the weight of $||f||_2$ is concentrated where $|S_1| + |S_2| + \ldots + |S_n| > c_4 \log n$ which is a contradiction if $c_4$ is sufficiently small.

\begin{quote}
**Alternative proof for Theorem 1 (sketch):** Let us assume again that $X = [0, 1]$ and that $f$ is monotone. The (ordinary) Fourier expansion of $f$ is:

$$f = \sum_{x \in \mathbb{Z}^n} \hat{f}(x) e^{2\pi i x \cdot \epsilon}.$$  

Define $\tilde{f}(k) = \sum_{x \in \mathbb{Z}^n} \hat{f}(k) |k|^{1/2}.$

Clearly $\tilde{f}(k)$ is non-negative and $\tilde{f}(f) = 0$ iff $f$ is a constant function.

**Lemma 3':** Let $f: X \to \{0, 1\}$ be a monotone function, then $\tilde{f}(f) \leq c$ for some absolute constant $c$.

**Proof:** Easy.

**Lemma 4':** Define $P_a = \sum x \theta_a e^{2\pi i x \cdot \epsilon}$. Then for $a > 0$ small enough, and for every $g: [0, 1] \to \mathbb{R}$, $\| P_a * g \|_2 \leq \|g\|_{4/3}$.

**Proof:** This follows easily by the Riesz interpolation theorem by showing that for $a > 0$ sufficiently small, $\| P_a * g \|_\infty \leq \|g\|_2$ and $\| P_a * g \|_1 \leq \|g\|_1$.

The proof of Theorem 1 proceeds as before: Just define

$$W_k = \sum_{x \in \mathbb{Z}^n} \hat{f}(x) e^{2\pi i x \cdot \epsilon},$$

(23)

and replace the operator $T_\epsilon$ by $g \to \hat{g}(\bigotimes \mathbb{Z}^n P_a) * g$.

**Remark:** In [KKL] stronger inequalities concerning the $L_p$-norms of the vector of influences $(I_f(1), \ldots, I_f(n))$ are proved, and some estimates on the absolute constants are given. Theorem 1 can be sharpened in a similar way. We omit the details.

**References**


RESIDUAL PROPERTIES OF FREE GROUPS, III

BY

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ABSTRACT

In this paper we want to prove the following theorem: Let $\mathcal{X}$ be an infinite set of non-abelian finite simple groups. Then the free group $F_2$ on 2 generators is residually $\mathcal{X}$. This answers a question first posed by W. Magnus and later by A. Lubotsky [9], Yu. Gorchakov and V. Levchuk [4].

1. Introduction

A group $G$ is called residually $\mathcal{X}$ if the intersection of all normal subgroups $N \triangleleft G$ such that $G/N \in \mathcal{X}$ is the trivial group. In this paper we consider a certain residual property of free groups $F_n$ on $n$ generators ($n \geq 2$). We consider the case in which every group in $\mathcal{X}$ is a non-abelian finite simple group and $\mathcal{X}$ is infinite. For these classes we prove the following theorem:

THEOREM 1: Let $\mathcal{X}$ be any infinite set of non-abelian finite simple groups. Then the free group $F_2$ on 2 generators is residually $\mathcal{X}$.

This answers a question first posed by W. Magnus and later by A. Lubotsky [9], Yu. Gorchakov and V. Levchuk [4]. As every non-abelian free group $F_n$ is residually $\{F_2\}$ [14], the transitivity implies that $F_n$ is also residually $\mathcal{X}$. So the theorem still holds for any free group $F_n$, where $n$ is a cardinal number greater than 1.

To prove Theorem 1 we show the following:

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