Hypertrees

Nati Linial

Uri Feige 60, January 2020
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We see that a graph is synonymous with a one-dimensional simplicial complex. It has zero-dimensional faces (aka vertices)
Familiar objects with new names

We see that a graph is synonymous with a one-dimensional simplicial complex. It has zero-dimensional faces (aka vertices) and one dimensional faces $=$ edges.
What is expressible in this language?

How do you say a tree in the language of simplicial complexes?
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Of course, a tree is a connected and acyclic graph.
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Does the language of simplicial complexes provide analogous terms?
ISRAEL JOURNAL OF MATHEMATICS, Vol. 45, No. 4, 1983

ENUMERATION OF Q-ACYCLIC
SIMPLICIAL COMPLEXES

BY
GIL KALAI

ABSTRACT
Let $\mathcal{C} = \mathcal{C}(n, k)$ be the class of all simplicial complexes $C$ over a fixed set of $n$ vertices ($2 \leq k \leq n$) such that: (1) $C$ has a complete $(k - 1)$-skeleton, (2) $C$ has precisely $\binom{n}{k-1}$ $k$-faces, (3) $H_k(C) = 0$. We prove that for $C \in \mathcal{C}$, $H_{k-1}(C)$ is a finite group, and our main result is:

$$\sum |H_{k-1}(C)|^2 = n \binom{n-2}{k}.$$
Recall the incidence matrix of a graph

\[ V \times E \quad \text{Vertices vs. edges.} \]

\[
A_G = \begin{pmatrix}
\vdots & \vdots & \ldots & ij & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \ldots & +1 & \ldots & \ldots & \ldots & \ldots \\
i & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \ldots & -1 & \ldots & \ldots & \ldots & \ldots \\
j & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]
Incidence matrices tell many things

- $G$ is connected iff $A_G$ has a trivial left kernel.

Because $A_G$'s left kernel is the linear span of the indicator vectors of $G$'s connected components.

The cycle space of $G$ is the right kernel of $A_G$.

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  - Because $A_G$’s right kernel is the linear span of the indicator vectors of $G$’s cycle.
Theorem

If $G = (V, E)$ is a graph with $n$ vertices and $n - 1$ edges, then TFAE

1. $G$ is connected.
2. $G$ is acyclic.
Recall: Equivalent descriptions of trees

**Theorem**

If $G = (V, E)$ is a graph with $n$ vertices and $n - 1$ edges, then TFAE

1. $G$ is connected.
2. $G$ is acyclic.
3. $G$ is collapsible.
Why $G$ is connected iff it is acyclic

For every $G$, $\text{rank}(A_G) \leq \text{rank}(A_{K_n}) = n - 1$.
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The only linear dependence among the rows is $1A_{K_n} = 0$.

1. $G$ is connected $\iff A_G$ has a trivial left kernel.
2. $G$ is acyclic $\iff A_G$ has a zero right kernel.
3. The $n - 1$ columns of $A_G$ are linearly independent.
Collapsibility

An elementary collapse is a step where you remove a vertex of degree one and the single edge that contains it.
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A graph $G$ is collapsible if by repeated application of elementary collapses you can eliminate all of the edges in $G$. 
Collapsing - a linear algebra perspective

Let $A_G$ be the incidence matrix of graph $G$. In an elementary collapse we erase row $i$ and column $e$ of $A_G$ where the $(i, e)$ entry is the only nonzero entry in the $i$-th row.

Recall: $e$ is the one and only edge incident with vertex $i$.

$G$ is collapsible if it is possible to eliminate all its columns by a series of elementary collapses. This implies that $G$ is acyclic - Collapsing yields a proof that the right kernel is empty.
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But note

As we saw, connectivity and acyclicity are linear algebraic. In contrast collapsibility is a purely combinatorial condition.
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As we saw, connectivity and acyclicity are linear algebraic. In contrast collapsibility is a purely combinatorial condition. Indeed we will soon see that in higher dimensions collapsibility implies connectivity and acyclicity, but the reverse implication does not hold.
We need a high-dimensional analog of the incidence matrix.
Boundary operators of simplicial complexes

$(d-1)$-dimensional faces vs. $d$-dimensional faces.

$$
\partial_2 = \\
\begin{pmatrix}
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ...

... & ... & ... & ... & ... & ... & ... & ... & ...

... & ... & ... & ... & ... & ... & ... & ... & ...

... & ... & ... & ... & ... & ... & ... & ... & ...

... & ... & ... & ... & ... & ... & ... & ... & ...

... & ... & ... & ... & ... & ... & ... & ... & ...

... & ... & ... & ... & ... & ... & ... & ... & ...

... & ... & ... & ... & ... & ... & ... & ... & ...

... & ... & ... & ... & ... & ... & ... & ... & ...

\end{pmatrix}
$$
Does this suggest what a hypertree is?

We know where to start:
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Q: What is the rank of $\partial_d$?
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Q: What is the rank of $\partial_d$?

A: $\binom{n-1}{d}$,
Does this suggest what a hypertree is?

We know where to start:

Q: What is the rank of $\partial_d$?

A: $\binom{n-1}{d}$, let’s prove it

That $\text{rank}(\partial_d) \leq \binom{n-1}{d}$ follows from $\partial_{d-1}\partial_d = 0$.

We will show that $\text{rank}(\partial_d) \geq \binom{n-1}{d}$ by exhibiting an explicit set of $\binom{n-1}{d}$ linearly independent columns, i.e., the set of $d$-faces of a $d$-dimensional hypertree.
So, what is a $d$-dimensional hypertree?

It is a $d$-dimensional simplicial complex with

- A full $(d-1)$-dimensional skeleton.
- It has $(n-1)d$ $d$-dimensional faces.
- Whose boundary operator $\partial_d$ has
  - a trivial left kernel.
  - zero right kernel.
  - The $(n-1)d$ columns of its $\partial_d$ span the columns of the matrix of all $(d-1)$-faces vs. all $d$-faces.
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Can you please show an example of hypertree?

Arguably the simplest one-dimensional (\(\equiv\)graphic) tree is a **star**, i.e., all 1-dimensional faces (\(\equiv\)edges) that contain, say, vertex \(n\).
Can you please show an example of hypertree?

Arguably the simplest one-dimensional (=graphic) tree is a star, i.e., all 1-dimensional faces (=edges) that contain, say, vertex $n$. The same works in every dimension: Take all $d$-faces (=sets of size $d + 1$) which contain the vertex $n$. Let’s see how this works.
Recall the boundary operator $\partial_d$ of the full $d$-dimensional $n$-vertex full complex. It is a $\binom{n}{d} \times \binom{n}{d+1}$ matrix of $\{0, -1, 1\}$.
Recall the boundary operator $\partial_d$ of the full $d$-dimensional $n$-vertex full complex. It is a $\binom{n}{d} \times \binom{n}{d+1}$ matrix of $\{0, -1, 1\}$. Rows in this matrix represent $(d-1)$-dimensional faces (=sets of size $d$). These sets fall in two categories:
I: Sets which do not contain the vertex \( n \), there are \( \binom{n-1}{d} \) of those.
The $d$-dimensional hyperstar (contd.)

I: Sets which do not contain the vertex $n$, there are \( \binom{n-1}{d} \) of those.

II: Those which do contain the vertex $n$, their number is: \( \binom{n-1}{d-1} \).

It is easily verified that the rows in category I linearly span those from category II. We can therefore eliminate the latter without losing in rank.
Its vertex set is $V = [n]$. Every subset of $V$ of size $\leq d$ is a face (its $(d - 1)$-skeleton is full). A set of size $d + 1$ is a $d$-dimensional face iff it contains the vertex $n$. 
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Its vertex set is $V = [n]$. Every subset of $V$ of size $\leq d$ is a face (its $(d-1)$-skeleton is full). A set of size $d + 1$ is a $d$-dimensional face iff it contains the vertex $n$. Why is this a hypertree? Because after category II rows are eliminated, what remains from the matrix $\partial_d$ is simply the identity matrix - clearly a full rank matrix. In the row corresponding to set $S$, the only nonzero entry is in the column corresponding to $S \dot{\cup} \{n\}$. 
Collapsibility in higher dimensions

Let $X$ be a $d$-dimensional complex.
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Let $X$ be a $d$-dimensional complex. If some $(d - 1)$-dimensional face $\tau$ is contained in a unique $d$-dimensional face $\sigma$, then the corresponding elementary collapse is to eliminate both $\tau$ and $\sigma$ from $X$. 

$X$ is $d$-collapsible if it is possible to eliminate all its $d$-faces by a series of elementary collapses. The linear algebra perspective of collapsibility shows that it implies acyclicity. But....
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A little surprise

\[ \binom{6-1}{2} = 10 \]

Figure: A triangulation of the projective plane
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This example shows (at least) two things: Unlike the 1-dimensional case of graphs, the definition of a $d$-dimensional hypertree depends on the underlying field.

Indeed: The 6-point triangulation of the projective plane is a $\mathbb{Q}$-hypertree, but not a $\mathbb{F}_2$-hypertree.
In dimension $d \geq 2$ $d$-collapsibility is a stronger condition than being a hypertree.
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**Conjecture**

*For every $d \geq 2$ and for every field $\mathbb{F}$ and $n \to \infty$ almost none of the $n$-vertex $d$-dimensional $\mathbb{F}$-hypertrees are collapsible.*

This remains open, and is supported by rigorous numerical experiments.
Q: Can you, at least, come up with more examples of non-collapsible hypertrees?
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A construction: Let $n$ be prime and $d \geq 2$. Fix a subset $A \subset \mathbb{Z}_n$ of cardinality $|A| = d + 1$. The sum complex $X_A$ corresponding to $A$ has a full $(d - 1)$-dimensional skeleton and contains a $d$-face $\sigma$ iff $\sum_{x \in \sigma} x \in A$. 

Theorem (L., Meshulam, Rosenthal) The complex $X_A$ is always a $Q$-hypertree. It is collapsible iff $A$ forms an arithmetic progression.
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Theorem (L., Meshulam, Rosenthal)

The complex $X_A$ is always a $\mathbb{Q}$-hypertree. It is collapsible iff $A$ forms an arithmetic progression.
Theorem (Cayley’s Formula, Borchardt 1860)

The number of trees with vertex set \([n]\) is \(n^{n-2}\).

Theorem (Kalai 1983)

\[
\sum_{T} |H_{d-1}(T)|^2 = n\binom{n-2}{d}
\]

where the sum is over all \(n\)-vertex \(d\)-dimensional hypertrees \(T\).
But how many $d$-hypertrees are there?

Open Problem

For $d \geq 2$ and large $n$, find (at least approximately) the number of $d$-dimensional $n$-vertex $\mathbb{Q}$-hypertrees.

Kalai’s Formula yields estimates, but falls short of an asymptotic formula. In joint work with Y. Peled these estimates were significantly improved, though a full answer still eludes us.
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Here is the strategy that we used to derive our bounds.
A random process

We consider the random process that starts with a full \((d - 1)\)-dimensional skeleton. At each step pick a random \(d\)-dimensional face \(\sigma \notin X\). If possible, we add \(\sigma\) to \(X\). Otherwise, we discard \(\sigma\).
A random process

We consider the random process that starts with a full $(d - 1)$-dimensional skeleton. At each step pick a random $d$-dimensional face $\sigma \not\in X$. If possible, we add $\sigma$ to $X$. Otherwise, we discard $\sigma$.

We cannot add $\sigma$ to $X$ iff this creates a new cycle. In this case we say that $\sigma$ is in the shade of $X$.

To wit: At each step we add to the current complex a random $d$-face $\sigma$ whose addition creates no new cycle ("$\sigma$ is not in the shade of $X$").
Let $G = (V, E)$ be a disconnected graph, and let $ij \notin E$. We say that $ij$ is in $G$’s shadow if $i$ and $j$ belong to the same connected component of $G$. 

Living in the shades
Let $G = (V, E)$ be a disconnected graph, and let $ij \notin E$. We say that $ij$ is in $G$’s shadow if $i$ and $j$ belong to the same connected component of $G$. In other words $ij$ is in $G$’s shadow iff the column corresponding to the edge $ij$ is in the linear span of the columns of $A_G$. 
Outside the shadow of an evolving graph
Shadow of an evolving 2-complex
Claim: If $X$ is an acyclic $d$-complex, then the number of $d$-faces in its shadow $Y$ is $\leq \frac{n\cdot|X|}{d+1}$
A lower bound on the number of hypertrees: A taste of the proof

**Claim:** If $X$ is an acyclic $d$-complex, then the number of $d$-faces in its shadow $Y$ is $\leq \frac{n \cdot |X|}{d+1}$

There are exactly $(d + 1) \cdot |Y|$ pairs $(v, \sigma)$ with $v$ a vertex in $\sigma$, a $d$-face in $Y$. Let $W$ be the set of $d$-faces in $Y$ that contain $v$. 

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There are exactly $(d + 1) \cdot |Y|$ pairs $(v, \sigma)$ with $v$ a vertex in $\sigma$, a $d$-face in $Y$. Let $W$ be the set of $d$-faces in $Y$ that contain $v$.

$W$ is an acyclic complex, being part of $v$’s hyperstar. So, the columns corresponding to $W$ are linearly independent and spanned by $X$. Therefore $|W| \leq |X|$, which proves our claim.
This is the grandfather of all models of random graphs. Introduced by Erdős and Rényi in the 60’s, a mainstay of modern combinatorics and still an important source of ideas and inspiration.
A little context - $G(n, p)$

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Closely related model: the evolution of random graphs starts with $n$ vertices and no edges. At each step add a random edge to the evolving graph.
A $d$-dimensional analog of $G(n, p)$

About 15 years ago, Roy Meshulam and I introduced a model of a random $d$-dimensional $n$-vertex complex $X_d(n, p)$. In dimension $d = 1$ the $X_1(n, p)$ model coincides with $G(n, p)$. 
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Start with a full $(d - 1)$-dimensional skeleton. (In the case of graphs - start with $n$ vertices.)

For each $d$-dimensional face $\sigma$, independently and with probability $p$, decide whether $\sigma \in X$. (For graphs - same with every edge.)
Some basic facts in $G(n, p)$ theory

Theorem (Erdős and Rényi ’60)

*The threshold for graph connectivity in $G(n, p)$ is*

\[ p = \frac{\ln n}{n} \]
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E.g., if $p \leq (1 - \epsilon)\frac{\ln n}{n}$, then whp a graph in $G(n, p)$ is disconnected.
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E.g., if $p \leq (1 - \epsilon)\frac{\ln n}{n}$, then whp a graph in $G(n, p)$ is disconnected.

Whereas if $p \geq (1 + \epsilon)\frac{\ln n}{n}$, whp a graph in $G(n, p)$ is connected.
The boundary operator $\partial_d$ has a trivial left kernel.

**Theorem (L. - Meshulam - Wallach)**

The threshold for connectivity of $X_d(n, p)$ is

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Specifically, whp, left kernel($\partial_d(X)$) is
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**Theorem (L. - Meshulam - Wallach)**

The threshold for connectivity of $X_d(n, p)$ is

$$p = \frac{d \ln n}{n}.$$ 

Specifically, whp, left kernel($\partial_d(X)$) is

- nontrivial for $p < (1 - \epsilon)\frac{d \ln n}{n}$, and
- trivial for $p > (1 + \epsilon)\frac{d \ln n}{n}$. 

Phase transition in $G(n, p)$ theory

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Start with $n$ isolated vertices and sequentially add a new random edge, one at a time. Initially every edge is isolated. Later, small and simple connected components appear.
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- small = cardinality $O(\log n)$.
- simple = a tree.
- Plus a Poisson number of unicyclic graphs with $O(\log n)$ vertices.
Crescendo - The phase transition

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Note: Time $\frac{n}{2}$ corresponds to $p = \frac{1}{n}$. 
In the wake of the revolution

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But if \( p > \frac{1+\epsilon}{n} \), \( G \) almost surely contains a cycle.
Meanwhile in high dimensions.....

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- Acyclic/cyclic: right kernel($\partial_d$) $\neq 0$. 

Recall: collapsible complexes are acyclic, so clearly $p_{\text{collapse}} \leq p_{\text{acyclic}}$. This inequality is strict.
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Concretely

Theorem (Lior Aronshtam, L., Tomasz Łuczak, Roy Meshulam, Yuval Peled)

The collapsibility threshold in $X_d(n, p)$ is

$$(1 + o_d(1))\frac{\log d}{n}.$$
Concretely

Theorem (Lior Aronshtam, L., Tomasz Łuczak, Roy Meshulam, Yuval Peled)

- The collapsibility threshold in $X_d(n, p)$ is

$$
(1 + o_d(1)) \frac{\log d}{n}.
$$

- The threshold for having a cycle \textit{whp} is

$$
d + 1 - o_d(1) \frac{1}{n}.
$$
Resolving the remaining major difficulty

We have no notion of a high-dimensional connected component

**Theorem (L., Y. Peled)**

Exactly at the same $p = \frac{c}{n}$ where $X_d(n, p)$ almost surely acquires a cycle, the shadow of the complex becomes gigantic ($\Omega(n^{d+1})$ faces).

This statement applies in all dimensions. However, when $d = 1$ the limit distribution is continuous but not smooth, while for $d \geq 2$ the limit distribution is discontinuous.
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A view of phase transition in $G(n, p)$
Phase transition in $X_2(n, p)$ complexes
More surprises in the shadows

Easy Observation

Let $G$ be an "almost tree", i.e., an $n$ vertex forest with $n - 2$ edges (and hence with two connected components). Then at least $(1 - o(1)) \frac{n^2}{4}$, i.e., at least half of the remaining edges, are in $G$'s shadow.
Construction: Let $X$ be a 2-dimensional $n$-vertex complex with a full 1-dimensional skeleton. The 2-faces of $X$ are the arithmetic triples of difference $\neq 1$. Easy fact: The number of 2-faces in $X$ is $\binom{n-1}{2} - 1$ (one less than a 2-dimensional hypertree).
Surprises in the shadows (contd.)

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Theorem (L., Newman, Peled, Rabinovich)

The complex $X$ is $\mathbb{Q}$-acyclic. Assuming the Riemann hypothesis\(^1\), there are infinitely many primes $n$ for which $X$ has an empty shadow.

\footnote{\textsuperscript{1}It actually suffices to assume the weaker Artin’s conjecture.}
Recent work of Anari, Liu, Oveis Gharan and Vinzant shows that a most natural algorithm for sampling hypertrees converges in polynomial time.
Hyperpaths?

A path is a tree where every vertex is in two or fewer edges.

Can we construct $d$-dimensional hypertree where every $(d - 1)$-dimensional face is contained in $d + 1$ or fewer $d$-dimensional face?
If so, how many are they? In dimension one: If $P_n$ denotes the number of $n$-vertex paths and $T_n$ is the number of $n$-vertex trees, then

$$\frac{P_n}{T_n} = \left(\frac{1}{e} + o(1)\right)^n$$

and in higher dimension?