Precoded Integer-Forcing Universally Achieves the MIMO Capacity to Within a Constant Gap

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Joint work with Uri Erez

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The MIMO Channel

$y = Hx + z$

- $H \in \mathbb{C}^{N \times M}$, $x \in \mathbb{C}^{M \times 1}$ and $z \sim \mathcal{CN}(0, I_N)$.
- Power constraint is $\mathbb{E}\|x\|^2 \leq M \cdot \text{SNR}$. 

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Precoded Integer-Forcing Equalization
The MIMO Channel

Closed-loop

\[ C = \max_{Q \succ 0 : \text{trace } Q \leq M \cdot \text{SNR}} \log \det \left( I + QH^\dagger H \right) \]
The MIMO Channel

Closed-loop

\[ C = \max_{Q > 0 : \text{trace } Q \leq M \cdot \text{SNR}} \log \det \left( I + QH^\dagger H \right) \]

Open-loop

Optimizing \( Q \) is impossible. Isotropic transmission \( Q = \text{SNR} \cdot I \) is a reasonable idea and gives

\[ C_{\text{WI}} = \log \det \left( I + \text{SNR}H^\dagger H \right) \]
The MIMO Channel

Closed-loop

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\[ C_{WI} = \log \det \left( I + \text{SNR} H^\dagger H \right) \]

Definition: Compound channel

The compound MIMO channel with capacity \( C_{WI} \) consists of the set of all channel matrices

\[ \mathcal{H}(C_{WI}) = \left\{ H \in \mathbb{C}^{N \times M} : \log \det \left( I + \text{SNR} H^\dagger H \right) = C_{WI} \right\} \]
How can we approach the compound channel capacity in practice?*

*practice = scalar AWGN coding & decoding + linear pre/post processing
Decoupling Decoding from Equalization

Transmitter

Encoder

\( w \)

\( x_1 \)

\( x_M \)

Channel

\( H \)

\( z_1 \)

\( y_1 \)

\( z_N \)

\( y_N \)

Receiver

Decoder

\( \hat{w} \)
Decoupling Decoding from Equalization

Split \( w \) to \( M \) messages \( w_1, \ldots, w_M \)

encode each message separately

equalize channel and decode each message separately
Closed-loop

Can transform the channel to a set of parallel SISO channels via SVD or QR

- Use standard AWGN encoders and decoders (e.g., turbo, LDPC) for the SISO channels
- Gap to capacity is the same as that of the AWGN codes
Compound channel

Much less is known...

- Can still apply QR at the receiver, but how should the transmitter allocate rates to the different streams?
- Can also apply linear equalization (ZF or MMSE), but loss can be large
The MIMO Channel - Practical Schemes

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Finding schemes with adequate performance guarantees for the compound channel is difficult
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The MIMO Channel - Practical Schemes

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**Less restricting benchmarks became common**

**Statistical approach**
The MIMO Channel - Practical Schemes

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Finding schemes with adequate performance guarantees for the compound channel is difficult

**Less restricting benchmarks became common**

\[ \mathbb{E}_H (P_e) = \mathbb{E}_{C_W I} (\mathbb{E}_H (P_e | C_W I)) \]
Diversity-multiplexing tradeoff (Zheng-Tse IT03)

- Introduced as a physical characterization of the channel
- Has become a benchmark for assessing practical coding schemes
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As a benchmark, DMT is powerful, but has two weaknesses:
The MIMO Channel - DMT

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As a benchmark, DMT is powerful, but has two weaknesses:

Weakness #1 - (lack of) robustness to channel statistics

- DMT optimality of a scheme does not translate to performance guarantees for specific channel realizations
  \[\Rightarrow\] Can design a scheme to work well only for typical channels

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Precoded Integer-Forcing Equalization
# The MIMO Channel - DMT

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### Weakness #1 - (lack of) robustness to channel statistics

- DMT optimality of a scheme does not translate to performance guarantees for specific channel realizations
  \[\Rightarrow\] Can design a scheme to work well only for typical channels

### Solution: approximately universal codes

- Introduced by Tavildar and Vishwanath (IT06)
- DMT optimal regardless of the channel statistics
The MIMO Channel - DMT

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- Introduced as a physical characterization of the channel
- Has become a benchmark for assessing practical coding schemes

As a benchmark, DMT is powerful, but has two weaknesses:

Weakness #2 - crude measure of error probability

- For “good” channel realizations, the error probability is only required to be smaller than the outage probability
  \[ \Rightarrow \] A scheme with short block length (essentially “uncoded”) can be DMT optimal
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When not in outage, we want communication to be reliable
This Work

A low-complexity scheme that achieves the compound MIMO capacity to within a constant gap.
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Constant gap-to-capacity also implies

- DMT optimality
- Constant gap to the outage capacity for any channel statistics
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Constant gap-to-capacity also implies

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Main result

IF equalization with space-time coded transmission can achieve any rate

\[ R < C_{WI} - \Gamma \left( \delta_{\min}(C_{\infty}^{ST}), M \right) \]

where \( \Gamma \left( \delta_{\min}(C_{\infty}^{ST}), M \right) \triangleq \log \frac{1}{\delta_{\min}(C_{\infty}^{ST})} + 3M \log(2M^2) \)
Precoded Integer-Forcing

For $2 \times 2$ Rayleigh fading with Golden code precoding

![Gap-to-capacity histogram at $C_{\text{WI}}=30$ bits](image)
Integer-Forcing: Background

Proposed by Zhan et al. ISIT2010
Antennas transmit independent streams (BLAST).
All streams are codewords from the same linear code with rate $R$. 
Rather than equalizing $\mathbf{H}$ to identity (as in ZF or MMSE), in IF the channel is equalized to a full-rank $\mathbf{A} \in \mathbb{Z}^M + i\mathbb{Z}^M$

$$
\mathbf{B} = \mathbf{A} \mathbf{H}^\dagger \left( \text{SNR}^{-1} \mathbf{I} + \mathbf{H} \mathbf{H}^\dagger \right)^{-1}
$$
A linear combination of codewords with integer coefficients is a codeword

\[ \tilde{y}_{\text{eff}, 1} = \sum_{m=1}^{M} a_{1m} x_m + z_{\text{eff}, 1} \]

\[ \tilde{y}_{\text{eff}, M} = \sum_{m=1}^{M} a_{Mm} x_m + z_{\text{eff}, M} \]

\( x_1 \in C \rightarrow v_1 \in C \rightarrow \tilde{y}_{\text{eff}, 1} \)

\( \vdots \)

\( x_M \in C \rightarrow v_M \in C \rightarrow \tilde{y}_{\text{eff}, M} \)

\( \Rightarrow \) Can decode the linear combinations - remove noise

\( \Rightarrow \) Can solve noiseless linear combinations for the transmitted streams
A linear combination of codewords with integer coefficients is a codeword

\[ \Rightarrow \text{Can decode the linear combinations - remove noise} \]

\[ \Rightarrow \text{Can solve noiseless linear combinations for the transmitted streams} \]
\( x_1 \in C \quad \vdots \quad x_M \in C \)

\[
\begin{align*}
\mathbf{v}_1 \in C &\quad \mathbf{A} &\quad \mathbf{v}_M \in C \\
\mathbf{y}_{\text{eff},1} &\quad = \sum_{m=1}^{M} a_{1m} x_m + z_{\text{eff},1} \\
\vdots &\quad \vdots \\
\mathbf{y}_{\text{eff},M} &\quad = \sum_{m=1}^{M} a_{Mm} x_m + z_{\text{eff},M}
\end{align*}
\]
**Integer-Forcing: Background**

\[ \mathbf{x}_1 \in \mathcal{C} \quad \vdash \quad \mathbf{v}_1 \in \mathcal{C} \quad \Downarrow \quad \mathbf{y}_{\text{eff},1} = \sum_{m=1}^{M} a_{1m} \mathbf{x}_m + \mathbf{z}_{\text{eff},1} \]

\[ \vdots \]

\[ \mathbf{x}_M \in \mathcal{C} \quad \vdash \quad \mathbf{v}_M \in \mathcal{C} \quad \Downarrow \quad \mathbf{y}_{\text{eff},M} = \sum_{m=1}^{M} a_{Mm} \mathbf{x}_m + \mathbf{z}_{\text{eff},M} \]

- Effective noise \( \mathbf{z}_{\text{eff},k} \) has effective variance

\[
\sigma_{\text{eff},k}^2 \triangleq \frac{1}{n} \mathbb{E} \| \mathbf{z}_{\text{eff},k} \|^2 = \text{SNR} \mathbf{a}_{k}^\dagger \left( \mathbf{I} + \text{SNR} \mathbf{H}^\dagger \mathbf{H} \right)^{-1} \mathbf{a}_k
\]

where \( \mathbf{a}_{k}^\dagger \) is the \( k \)th row of \( \mathbf{A} \).
Integer-Forcing: Background

\[ \begin{align*}
\mathbf{x}_1 & \in \mathcal{C} \\
\vdots & \\
\mathbf{x}_M & \in \mathcal{C} \\
\end{align*} \]

\[ \begin{align*}
\mathbf{z}_{\text{eff},1} \\
\vdots \\
\mathbf{z}_{\text{eff},M} \\
\end{align*} \]

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\tilde{\mathbf{y}}_{\text{eff},1} & = \sum_{m=1}^{M} a_{1m} \mathbf{x}_m + \mathbf{z}_{\text{eff},1} \\
\vdots & \\
\tilde{\mathbf{y}}_{\text{eff},M} & = \sum_{m=1}^{M} a_{Mm} \mathbf{x}_m + \mathbf{z}_{\text{eff},M} \\
\end{align*} \]

- Same codebook used over all subchannels
  \[ \implies \text{the subchannel with the largest noise dictates the performance} \]

\[ \begin{align*}
\text{SNR}_{\text{eff},k} & \triangleq \frac{\text{SNR}}{\sigma_{\text{eff},k}^2} = \left[ a_k^\dagger \left( \mathbf{I} + \text{SNR} \mathbf{H}^\dagger \mathbf{H} \right)^{-1} a_k \right]^{-1} \\
\text{SNR}_{\text{eff}} & \triangleq \min_{k=1,\ldots,M} \text{SNR}_{\text{eff},k} = \left[ \max_{k=1,\ldots,M} \left. a_k^\dagger \left( \mathbf{I} + \text{SNR} \mathbf{H}^\dagger \mathbf{H} \right)^{-1} a_k \right. \right]^{-1}
\end{align*} \]
For AWGN capacity achieving nested lattice codebook $C$

$$R_{IF} < M \log(SNR_{eff})$$
Integer-Forcing: Background

For AWGN capacity achieving nested lattice codebook $C$

$$R_{IF} < M \log(\text{SNR}_{\text{eff}})$$

To approach $C_{WI}$ we need $\text{SNR}_{\text{eff}} \approx 2^{\frac{C_{WI}}{M}}$
Integer-Forcing: SNR_{eff}

\[
\text{SNR}_{\text{eff}} = \frac{1}{\min_{A \in \mathbb{Z}^{M \times M} + i\mathbb{Z}^{M \times M}} \max_{k=1,\ldots,M} a_k^\dagger (I + \text{SNR} H^\dagger H)^{-1} a_k}
\]
Integer-Forcing: $\text{SNR}_{\text{eff}}$

$$\text{SNR}_{\text{eff}} = \frac{1}{\min_{A \in \mathbb{Z}^{M \times M} + i\mathbb{Z}^{M \times M}} \max_{k=1, \ldots, M} a_k \dagger (I + \text{SNR} H \dagger H)^{-1} a_k} \quad \text{with } \det(A) \neq 0$$

- Does not give much insight to the dependence on $H$ 😞
Integer-Forcing: $\text{SNR}_{\text{eff}}$

$$\text{SNR}_{\text{eff}} = \frac{1}{\min_{A \in \mathbb{Z}^{M \times M} + i \mathbb{Z}^{M \times M}} \max_{k=1, \ldots, M} a_k^\dagger (I + \text{SNR} H^\dagger H)^{-1} a_k}$$

- Does not give much insight to the dependence on $H$ 😞

- Fortunately, using a transference theorem by Banaszczyk we can lower bound with a simple expression 😊
Theorem - \( SNR_{\text{eff}} \) bound

\[
SNR_{\text{eff}} > \frac{1}{4M^2} \min_{a \in \mathbb{Z}^M + i \mathbb{Z}^M \setminus 0} a^\dagger \left( I + SNR H^\dagger H \right) a
\]
Theorem - SNR\textsubscript{eff} bound

\[
\text{SNR}_{\text{eff}} > \frac{1}{4M^2} \min_{a \in \mathbb{Z}_M^M + i\mathbb{Z}_M^M \setminus 0} a^\dagger \left( I + \text{SNR} H^\dagger H \right) a
\]

Let

\[
\text{QAM}(L) \triangleq \{-L, -L + 1, \ldots, L - 1, L\} + i\{-L, -L + 1, \ldots, L - 1, L\},
\]

and define

\[
d_{\min}(H, L) \triangleq \min_{a \in \text{QAM}^M(L) \setminus 0} \| Ha \|
\]
Theorem - SNR\textsubscript{eff} bound

\[ \text{SNR}_{\text{eff}} > \frac{1}{4M^2} \min_{a \in \mathbb{Z}^M + i\mathbb{Z}^M \setminus 0} a^\dagger \left( I + \text{SNR} H^\dagger H \right) a \]

Let

\[ \text{QAM}(L) \triangleq \{-L, -L + 1, \ldots, L - 1, L\} + i \{-L, -L + 1, \ldots, L - 1, L\}, \]

and define \( d_{\text{min}}(H, L) \triangleq \min_{a \in \text{QAM}^M(L) \setminus 0} \| Ha \| \)

Corollary

\[ \text{SNR}_{\text{eff}} > \frac{1}{4M^2} \min_{L=1,2,\ldots} (L^2 + \text{SNR} d_{\text{min}}^2(H, L)) \]
Integer-Forcing: $\text{SNR}_{\text{eff}}$ via Uncoded $d_{\text{min}}$

$$\text{SNR}_{\text{eff}} > \frac{1}{4M^2} \min_{L=1,2,...} \left( L^2 + \text{SNR}d_{\text{min}}^2(H, L) \right)$$
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What can we guarantee for a specific channel realization?

Unfortunately nothing...
Integer-Forcing: $\text{SNR}_{\text{eff}}$ via Uncoded $d_{\text{min}}$

$$\text{SNR}_{\text{eff}} > \frac{1}{4M^2} \min_{L=1,2,...} \left( L^2 + \text{SNR}d_{\text{min}}^2(H, L) \right)$$

Example for a bad channel

$$H = \begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix}$$

- $\text{SNR}_{\text{eff}} = 1$, $R_{\text{IF}} = M \log(\text{SNR}_{\text{eff}}) = 0$.
- $C_{\text{WI}} - R_{\text{IF}}$ is unbounded (as with any BLAST scheme).
Integer-Forcing: $SNR_{\text{eff}}$ via Uncoded $d_{\text{min}}$

$$SNR_{\text{eff}} > \frac{1}{4M^2} \min_{L=1,2,...} \left( L^2 + SNRd_{\text{min}}^2(H, L) \right)$$

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Need to precode over time for transmit diversity
Instead of transmitting $M$ independent streams of length $n$ over $n$ time slots, transmit $MT$ independent streams over $nT$ time slots.

Before transmission, precode all $MT$ streams using a unitary matrix $P \in \mathbb{C}^{MT \times MT}$.  

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Space-Time Coding/Modulation

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Precoded Integer-Forcing

\[
\tilde{y} = \begin{bmatrix}
H & 0 & \cdots & 0 \\
0 & H & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H 
\end{bmatrix}
\]

\[P\tilde{x} + \tilde{z} = \mathcal{H}P\tilde{x} + \tilde{z} = \tilde{H}\tilde{x} + \tilde{z}\]

Can apply IF equalization to the aggregate channel [Domanovitz and Erez IEEEI12]
Precoded Integer-Forcing Equalization

\[ \tilde{y} = \begin{bmatrix} \mathbf{H} & 0 & \cdots & 0 \\ 0 & \mathbf{H} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H} \end{bmatrix} \]

Can apply IF equalization to the aggregate channel [Domanovitz and Erez IEEE112]

But how to choose \( \mathbf{P} \) to guarantee good performance?

- Large minimum distance for QAM translates to large SNR\(_{eff} \) for IF
- \( \mathbf{P} \) should maximize \( d_{\text{min}}^2(\mathcal{H}\mathbf{P}, L) \) for the worst-case matrix \( \mathbf{H} \)
- This problem was extensively studied under the linear dispersion space-time coding framework
- “Perfect” linear dispersion codes guarantee that \( d_{\text{min}}^2(\mathcal{H}\mathbf{P}, L) \) grows appropriately with \( C_{\text{WI}} \)
Proving the Lower Bound

**Theorem**

If \( \mathbf{P} \) generates a perfect linear dispersion code

\[
\text{SNR}_d^2(\mathcal{H}\mathbf{P}, L) \geq \left[ \delta_{\min}(C_{\infty}^{\text{ST}}) \frac{1}{M} 2^{\frac{C_{\text{WI}}}{M}} - 2M^2L^2 \right]^+
\]

for all channels matrices \( \mathbf{H} \)

Proof follows by using the properties of perfect codes and extending Tavildar and Vishwanath’s proof for the approximate universality criterion.
Proving the Lower Bound

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for all channels matrices \( \mathcal{H} \)

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**Combining with the SNR_{eff} lower bound**

For precoded IF with a generating matrix \( \mathbf{P} \) of a perfect ST “code”

\[
\text{SNR}_{\text{eff}} \geq \frac{1}{4M^4} \min_{L=1,2,...} \left( L^2 + \text{SNR}_{d_{\min}^2}(\mathcal{H}\mathbf{P}, L) \right) 
\]

\[
\geq \frac{1}{8M^6} \delta_{\min}(C^{ST}) \frac{1}{M} 2 \frac{C_{\text{WI}}}{M}
\]
Proving the Lower Bound

Since $R_{IF} = M \log(\text{SNR}_{\text{eff}})$ we get the main result.

For precoded IF with a generating matrix $\mathbf{P}$ of a perfect ST “code”

$$R_{IF} = M \log(\text{SNR}_{\text{eff}}) > C_{\text{WI}} - \Gamma \left( \delta_{\min}(C^{\text{ST}}), M \right)$$

where $\Gamma \left( \delta_{\min}(C^{\text{ST}}), M \right) \triangleq \log \frac{1}{\delta_{\min}(C^{\text{ST}})} + 3M \log(2M^2)$.
Since $R_{IF} = M \log(\text{SNR}_{\text{eff}})$ we get the main result

For precoded IF with a generating matrix $\mathbf{P}$ of a perfect ST “code”

$$R_{IF} = M \log(\text{SNR}_{\text{eff}}) > C_{WI} - \Gamma \left( \delta_{\text{min}}(C^S_T), M \right)$$

where $\Gamma \left( \delta_{\text{min}}(C^S_T), M \right) \triangleq \log \frac{1}{\delta_{\text{min}}(C^S_T)} + 3M \log(2M^2)$

Thanks for your attention!