Online Prediction, Low Regret, and Convex Duality

Shai Shalev-Shwartz

Toyota Technological Institute at Chicago

GIF Workshop, Tübingen, May 15-16, 2008

Predicting the next element of a binary sequence

Prediction task

For $t = 1, 2, \dots, T$

- Predict: $\hat{y}_t \in \{\pm 1\}$
- Get: $y_t \in \{\pm 1\}$
- Suffer loss: $\ell_{0-1}(\hat{y}_t, y_t) = \begin{cases} 1 & y_t \neq \hat{y}_t \\ 0 & y_t = \hat{y}_t \end{cases}$

Regret

- Best in hindsight $y^\star = \mathrm{sign}(\sum_t y_t)$
- Regret: $R_T \; = \sum_{t=1}^{T} \ell_{0-1}(\hat{y}_t, y_t) - \sum_{t=1}^{T} \ell_{0-1}(y^\star, y_t)$

Abstract Prediction Model

- Set of decisions S
- Set of loss functions $\mathcal{L} = \{\ell : S \to \mathbb{R}\}$

Prediction Game

For $t = 1, \ldots, T$

- Learner chooses a decision $\mathbf{w}_t \in S$
- Environment chooses a loss function $\ell_t \in \mathcal{L}$
- Learner suffers loss $\ell_t(\mathbf{w}_t)$

- Goal: Conditions on S and $\mathcal{L}$ that guarantee low regret

$$R_T \;:=\; \sum_{t=1}^{T} \ell_t(\mathbf{w}_t) - \sum_{t=1}^{T} \ell_t(\mathbf{w}^\star) \;\stackrel{!}{=}\; o(T)$$

Outline

- Identifying sufficient conditions for predictability
  - Size matters?
  - No !
  - Maybe yes with randomization ?
  - A modern view: revealing an underlying convexity
- Regret and Convex Duality
- Generality and related work
- Experimental results

Impossibility Result

- $S = \{\pm 1\}$
- $\mathcal{L} = \{\ell_{0-1}(w_t, 1)\ ,\ \ell_{0-1}(w_t, -1)\}$
- Adversary can make the cumulative loss of the learner to be T by using $\ell_t(\cdot) = \ell_{0-1}(\cdot, -w_t)$
- The loss of the constant prediction $w^\star = \mathrm{sign}(\sum_t w_t)$ is at most $T/2$
- Regret is at least $T/2$

Conclusion

- In the above example, $|S| = |\mathcal{L}| = 2$.
- Small size does not guarantee low regret

Solution: Randomized Predictions

- Learner predicts $\hat{y}_t = 1$ with probability w_t
- Best in hindsight: $y_t^\star = 1$ with probability $w^\star$ where $w^\star = \frac{|\{t : y_t = 1\}|}{T}$
- Analyze the expected regret:

$$\sum_{t=1}^{T} \mathbb{E}[\hat{y}_t \neq y_t] - \sum_{t=1}^{T} \mathbb{E}[y_t^\star \neq y_t]$$

- There are algorithms that achieve expected regret of $O(\sqrt{T})$

A modern view: revealing an underlying convexity

- Expected zero-one loss can be rewritten as

$$\mathbb{E}[\hat{y}_t \neq y_t] \;=\; \begin{cases} 1 - w_t & \text{if } \; y_t = 1 \\ w_t & \text{if } \; y_t = -1 \end{cases}$$

- Going back to our abstract model, we get that:
  - $S = [0, 1]$
  - $\mathcal{L} = \{\ell(w) = 1 - w \;,\; \ell(w) = w\}$

Properties

- All functions in $\mathcal{L}$ are linear (and thus are convex and Lipschitz)
- S is convex and bounded
- Sufficient conditions for low regret ?

Are we just playing with formalities ?

The convexity assumption is natural in many cases.

Example: Prediction with Expert Advice

- Learner receives a vector $(x_1^t, \ldots, x_d^t) \in [-1, 1]^d$ of experts advice
- Learner needs to predict a target $\hat{y}_t \in \mathbb{R}$
- Environment gives correct target $y_t \in \mathbb{R}$
- Learner suffers loss $|y_t - \hat{y}_t|$
- Goal: Be almost as good as the best experts committee
  $\sum_t |y_t - \hat{y}_t| - \sum_t |y_t - \langle \mathbf{w}^\star, \mathbf{x}^t \rangle| \stackrel{!}{=} o(T)$

Are we just playing with formalities ?

The convexity assumption is natural in many cases.

Example: Prediction with Expert Advice

- Learner receives a vector $(x_1^t, \ldots, x_d^t) \in [-1, 1]^d$ of experts advice
- Learner needs to predict a target $\hat{y}_t \in \mathbb{R}$
- Environment gives correct target $y_t \in \mathbb{R}$
- Learner suffers loss $|y_t - \hat{y}_t|$
- Goal: Be almost as good as the best experts committee

  $\sum_t |y_t - \hat{y}_t| - \sum_t |y_t - \langle \mathbf{w}^\star, \mathbf{x}^t \rangle| \overset{!}{=} o(T)$

Modeling

- S is the d-dimensional probability simplex
- $\mathcal{L} = \{\ell_{\mathbf{x},y}(\mathbf{w}) = |y - \langle \mathbf{w}, \mathbf{x} \rangle| \; : \; \mathbf{x} \in [-1, 1]^d, y \in [-1, 1]\}$

Are we just playing with formalities ?

Example: Convexifying finite decision sets

- Learner should predict an element $s_t \in S' = \{1, \ldots, N\}$
- Environment presents non-convex loss function $\ell'_t : S' \to [0, 1]$
- Learner suffers loss $\ell'_t(s_t)$
- Goal: Be almost as good as the best pure prediction
  $\sum_t \ell'_t(s_t) - \sum_t \ell'_t(s^\star) \stackrel{!}{=} o(T)$

Are we just playing with formalities ?

Example: Convexifying finite decision sets

- Learner should predict an element $s_t \in S' = \{1, \ldots, N\}$
- Environment presents non-convex loss function $\ell'_t : S' \to [0, 1]$
- Learner suffers loss $\ell'_t(s_t)$
- Goal: Be almost as good as the best pure prediction
  $\sum_t \ell'_t(s_t) - \sum_t \ell'_t(s^\star) \stackrel{!}{=} o(T)$

Modeling

- S is the N-dimensional probability simplex
- Prediction s_t is chosen randomly according to $\mathbf{w}_t \in S$
- $\mathcal{L} = \{\ell_{\mathbf{r}}(\mathbf{w}) = \langle \mathbf{w}, \mathbf{r} \rangle \; : \; \mathbf{r} \in [0, 1]^N\}$
- $\mathbb{E}[\ell'_t(s_t)] = \ell_{\ell'_t}(\mathbf{w}_t)$

Sufficient Conditions for low regret

The Online Convex Programming (OCP) model

- All functions in $\mathcal{L}$ are convex and L-Lipschitz
- S is convex and $\max\{\|\mathbf{w}\|_2 : \mathbf{w} \in S\} = D$
- Then, there exists an algorithm with regret $O(L\,D\,\sqrt{T})$
- This is tight (i.e. the minimax value of the game)

Bibliography

- The OCP model was presented by Gordon (1999)
- Zinkevich (2003) proved a regret bound of $O((L^2 + D^2)\sqrt{T})$

Dimension independency ?

Yes !

- The regret bound does not depend on the dimensionality of S
- Similarly to Support Vector Machines, we can use Kernel functions

Dimension independency ?

Yes !

- The regret bound does not depend on the dimensionality of S
- Similarly to Support Vector Machines, we can use Kernel functions

Yes ?

- Consider again the prediction with expert advice problem
- d experts, each of which gives an "advice" in $[-1, 1]$
- S is the probability simplex and thus $D = 1$
- Lipschitz constant is $L = \sqrt{d}$
- Regret is $\Omega(\sqrt{d\,T})$.
- Is this the best we can do ?

Low regret algorithmic framework for OCP

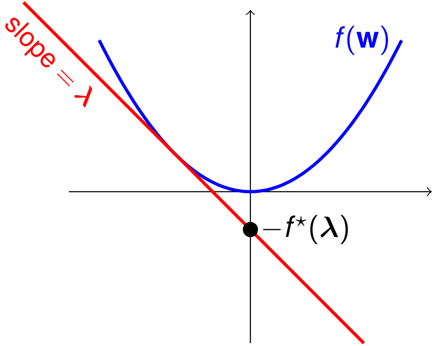
- A low regret algorithmic framework for OCP
- Family of sufficient conditions for low regret
- In particular – Alternatives to the Lipschitz condition
- In the expert committee example – logarithmic dependence on dimension
- Main observation: Relating regret and duality

Fenchel Conjugate

The Fenchel conjugate of the function $f : S \to \mathbb{R}$ is $f^\star : \mathbb{R}^d \to \mathbb{R}$

$$f^\star(\boldsymbol{\lambda}) \;=\; \max_{\mathbf{w} \in S} \langle \mathbf{w}, \boldsymbol{\lambda} \rangle - f(\mathbf{w})$$

If f is closed and convex then $f^{\star\star} = f$

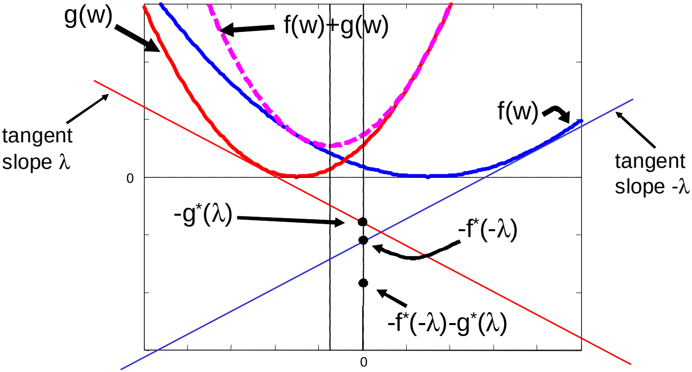


Fenchel Duality

$$\max_{\boldsymbol{\lambda}} \; -f^\star(-\boldsymbol{\lambda}) - g^\star(\boldsymbol{\lambda}) \quad \le \quad \min_{\mathbf{w}} \; f(\mathbf{w}) + g(\mathbf{w})$$

Fenchel Duality

$$\max_{\boldsymbol{\lambda}} \; -f^\star(-\boldsymbol{\lambda}) - g^\star(\boldsymbol{\lambda}) \;\; \leq \;\; \min_{\mathbf{w}} \; f(\mathbf{w}) + g(\mathbf{w})$$



Regret and Duality

- Recall that our goal is:

$$\forall \mathbf{w}^\star \in S, \quad \sum_{t=1}^{T} \ell_t(\mathbf{w}_t) - \sum_{t=1}^{T} \ell_t(\mathbf{w}^\star) \ \leq \ L\,D\sqrt{T}$$

Regret and Duality

- Recall that our goal is:

$$\forall \mathbf{w}^\star \in S, \quad \sum_{t=1}^{T} \ell_t(\mathbf{w}_t) - \sum_{t=1}^{T} \ell_t(\mathbf{w}^\star) \;\le\; L\,D\,\sqrt{T}$$

- Rewrite it in a 'silly' way

$$\sum_{t=1}^{T} \ell_t(\mathbf{w}_t) \;\le\; \min_{\mathbf{w}\in S}\; L\,D\,\sqrt{T} + \sum_{t=1}^{T} \ell_t(\mathbf{w})$$

Regret and Duality

- Recall that our goal is:

$$\forall \mathbf{w}^\star \in S, \quad \sum_{t=1}^{T} \ell_t(\mathbf{w}_t) - \sum_{t=1}^{T} \ell_t(\mathbf{w}^\star) \ \leq \ L\,D\sqrt{T}$$

- Rewrite it in a 'silly' way

$$\sum_{t=1}^{T} \ell_t(\mathbf{w}_t) \ \leq \ \min_{\mathbf{w} \in S} \ L\,D\sqrt{T} + \sum_{t=1}^{T} \ell_t(\mathbf{w})$$

- Replace $LD\sqrt{T}$ with a function $f : S \to \mathbb{R}$ s.t. $\max_{\mathbf{w}} f(\mathbf{w}) \leq LD\sqrt{T}$. E.g. $f(\mathbf{w}) = c\,\|\mathbf{w}\|^2$ for $c = L\sqrt{T}/D$. Obtaining:

$$\sum_{t=1}^{T} \ell_t(\mathbf{w}_t) \ \leq \ \min_{\mathbf{w} \in S} \ f(\mathbf{w}) + \sum_{t=1}^{T} \ell_t(\mathbf{w})$$

Regret and Duality

- Recall that our goal is:

$$\forall \mathbf{w}^\star \in S, \quad \sum_{t=1}^{T} \ell_t(\mathbf{w}_t) - \sum_{t=1}^{T} \ell_t(\mathbf{w}^\star) \;\leq\; L\,D\sqrt{T}$$

- Rewrite it in a 'silly' way

$$\sum_{t=1}^{T} \ell_t(\mathbf{w}_t) \;\leq\; \min_{\mathbf{w}\in S} \; L\,D\sqrt{T} + \sum_{t=1}^{T} \ell_t(\mathbf{w})$$

- Replace $LD\sqrt{T}$ with a function $f : S \to \mathbb{R}$ s.t. $\max_{\mathbf{w}} f(\mathbf{w}) \leq LD\sqrt{T}$. E.g. $f(\mathbf{w}) = c\,\|\mathbf{w}\|^2$ for $c = L\sqrt{T}/D$. Obtaining:

$$\sum_{t=1}^{T} \ell_t(\mathbf{w}_t) \;\leq\; \min_{\mathbf{w}\in S} \; f(\mathbf{w}) + \sum_{t=1}^{T} \ell_t(\mathbf{w})$$

- Lower bound of a minimization problem. Duality ?

Properties of the dual problem

$$\max_{\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_T} \; -f^\star\Big(-\sum_t \boldsymbol{\lambda}_t\Big) - \sum_t \ell_t^\star(\boldsymbol{\lambda}_t) \quad \leq \quad \min_{\mathbf{w}\in S} \; f(\mathbf{w}) + \sum_{t=1}^{T} \ell_t(\mathbf{w})$$

Decomposability of the dual

- There's a different dual variable for each online round
- Future loss functions do not affect dual variables of current and past rounds
- Therefore, the dual can be optimized incrementally
- To optimize $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_t$, it is enough to know $\ell_1, \ldots, \ell_t$

Primal-Dual Online Prediction Strategy

Algorithmic Framework

- Initialize $\boldsymbol{\lambda}_1 = \ldots = \boldsymbol{\lambda}_T = \mathbf{0}$
- For $t = 1, 2, \ldots, T$
  - Construct $\mathbf{w}_t$ from the dual variables
  - Receive ℓ_t
  - Update dual variables $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_t$

Primal-Dual Online Prediction Strategy

Algorithmic Framework

- Initialize $\boldsymbol{\lambda}_1 = \ldots = \boldsymbol{\lambda}_T = \mathbf{0}$
- For $t = 1, 2, \ldots, T$
  - Construct $\mathbf{w}_t$ from the dual variables
  - Receive ℓ_t
  - Update dual variables $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_t$

Lemma

Let $\mathcal{D}_t$ be the dual value at round t and w.l.o.g assume $\mathcal{D}_1 = 0$.

- *Assume that* $\max_{\mathbf{w} \in S} f(\mathbf{w}) \leq a\sqrt{T}$
- *Assume that* $\mathcal{D}_{t+1} - \mathcal{D}_t \geq \ell_t(\mathbf{w}_t) - \frac{a}{\sqrt{T}}$

Then, the regret is bounded by $2a\sqrt{T}$

The proof follows directly from the weak duality theorem

Strong convexity and sufficient dual increase

Strong Convexity w.r.t. norm

A function f is σ-strongly convex over S w.r.t $\|\cdot\|$ if for all $\mathbf{u}, \mathbf{v} \in S$

$$\frac{f(\mathbf{u})+f(\mathbf{v})}{2} \;\geq\; f(\tfrac{\mathbf{u}+\mathbf{v}}{2}) + \tfrac{\sigma}{8}\|\mathbf{u}-\mathbf{v}\|^2$$

Lemma (Sufficient Dual Increase)

Assume:

- *f is σ-strongly convex w.r.t. $\|\cdot\|$*
- *ℓ_t is closed and convex*
- *∇_t is a sub-gradient of ℓ_t at $\mathbf{w}_t$*

Then, there exists a simple dual update rule s.t.

$$\mathcal{D}_{t+1} - \mathcal{D}_t \;\geq\; \ell_t(\mathbf{w}_t) - \frac{\|\nabla_t\|_\star^2}{2\sigma}$$

Generalized Boundedness-Lipschitz condition

Theorem

Assume:

- *Exists* $f : S \to \mathbb{R}$ *which is* 1*-strongly convex w.r.t.* $\|\cdot\|$
- $D = \max_{\mathbf{w} \in S} \sqrt{f(\mathbf{w})}$
- ℓ_t *is closed and convex*
- $\|\nabla_t\|_\star \le L$ *(Lipschitz w.r.t. norm* $\|\cdot\|_\star$*)*

Then, there exists an algorithm with regret bound $2\,D\,L\,\sqrt{T}$

Generalized Boundedness-Lipschitz condition

Theorem

Assume:

- *Exists* $f : S \rightarrow \mathbb{R}$ *which is* 1*-strongly convex w.r.t.* $\|\cdot\|$
- $D = \max_{\mathbf{w} \in S} \sqrt{f(\mathbf{w})}$
- ℓ_t *is closed and convex*
- $\|\nabla_t\|_\star \leq L$ *(Lipschitz w.r.t. norm* $\|\cdot\|_\star$*)*

Then, there exists an algorithm with regret bound $2\,D\,L\,\sqrt{T}$

Example usage – back to expert problem

- Take f to be the relative entropy
- f is strongly convex w.r.t. $\|\cdot\|_1$ and $D = \sqrt{\log(d)}$
- $\|\nabla_t\|_\star = \|\mathbf{x}^t\|_\infty \leq 1$
- Regret bound becomes $O(\sqrt{\log(d)\,T})$

Self Boundedness instead of Lipschitz

Theorem

Replacing Lipschitz condition with the following self-bounded property:

$$\|\nabla_t\| \le L\,\sqrt{\ell_t(\mathbf{w}_t)}$$

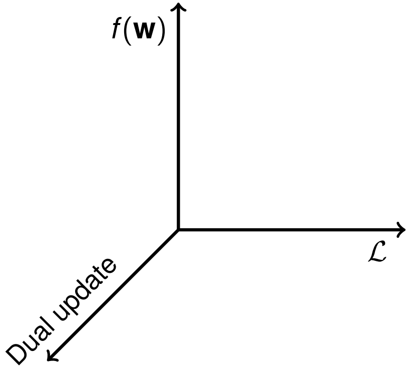
Then,

$$R_T \;\le\; O\left(L\,D\,\sqrt{\sum_t \ell_t(\mathbf{w}^\star)} + L^2\,D^2\right) \;.$$

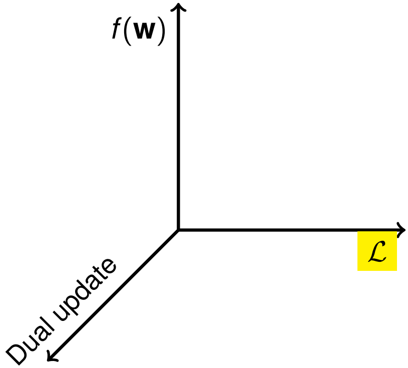
Examples

- $\ell(\mathbf{w}) = \frac{1}{2}(\langle \mathbf{w}, \mathbf{x}\rangle - y)^2$ is $(\sqrt{2}\|\mathbf{x}\|)$-self-bounded
- $\ell(\mathbf{w}) = \log(1.26 + \exp(-y\langle \mathbf{w}, \mathbf{x}\rangle))$ is $(\|\mathbf{x}\|)$-self-bounded

Generality and Related Work

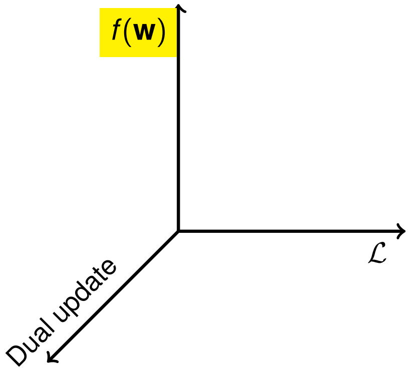


Generality and Related Work



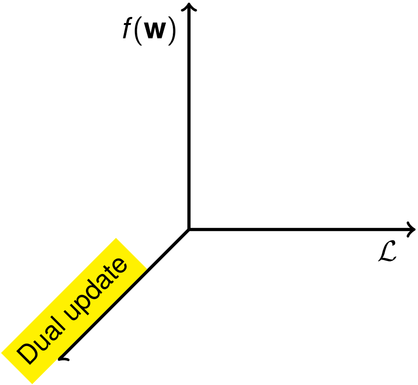
- Family of loss functions ($\mathcal{L}$)
  - Online Learning (Perceptron, linear regression, multiclass prediction, structured output, ...)
  - Game theory (Playing repeated games, correlated equilibrium)
  - Information theory (Prediction of individual sequences)
  - Convex optimization (SGD, dual decomposition)

Generality and Related Work



- Complexity function (f)
  - Online learning (Grove, Littlestone, Schuurmans; Kivinen, Warmuth; Gentile; Vovk)
  - Game theory (Hart and Mas-collel)
  - Optimization (Nemirovsky, Yudin; Beck, Teboulle, Nesterov)
  - Unified frameworks (Cesa-Bianchi and Lugosi)

Generality and Related Work



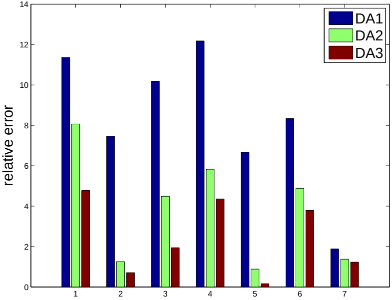
- Dual update schemes
  - Only two extremes were studied:
    - Gradient update (naive update of a single dual variable)
    - Follow the leader (Equivalent to full optimization)
  - Our analysis enables the entire spectrum

Experiments

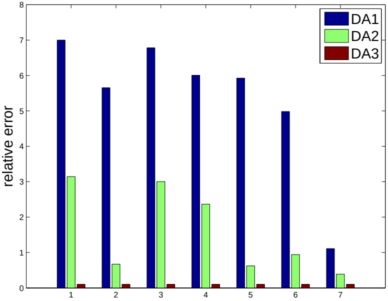
- Task: route emails to folders
- 7 users from the Enron dataset
- Bag of words representation
- 6 Algorithms
  - 2 complexity functions (Euclidean and Entropy)
  - 3 dual ascent methods
    - DA1: Fixed sub-gradient ($\boldsymbol{\lambda}_t = s_t \in \partial \ell_t(\mathbf{w}_t)$)
    - DA2: Optimal sub-gradient ($\boldsymbol{\lambda}_t = \alpha_t s_t$ with optimal α_t)
    - DA3: Optimal ($\boldsymbol{\lambda}_t = \arg\max_{\boldsymbol{\lambda}} \mathcal{D}(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{t-1}, \boldsymbol{\lambda}, 0, \ldots)$)
- Performance expectation
  - Entropy outperforms Euclidean
  - DA3 better than DA2 better than DA1

Experimental Results – 3 Dual Updates

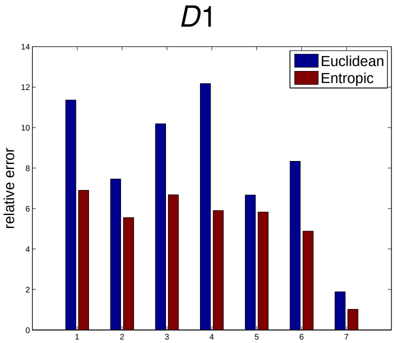
Euclidean complexity

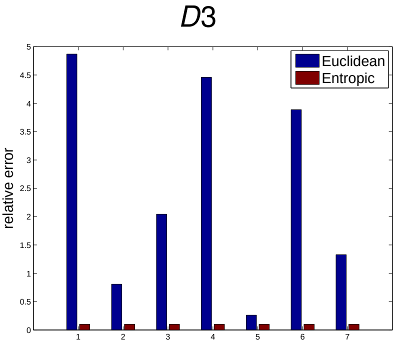


Entropic complexity



Experimental Results – 2 Complexity Functions





Summary and Future work

Summary

- The online convex programming is a powerful model
- Achieving low regret by primal-dual algorithmic framework
- Sufficient conditions for predictability

Current and future work

- Logarithmic regret algorithms
- Prediction with limited feedback (Bandit algorithms)
- Boosting, sparsity, and ℓ_1 norm
- Similar sufficient conditions for stochastic optimization (PAC learning)